turns out to be

$$\widetilde{C} = [\mathbf{0} \;\; \boldsymbol{I}] \begin{bmatrix} \widetilde{\boldsymbol{V}} & \widetilde{\boldsymbol{X}} \\ \widetilde{\boldsymbol{X}}' & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix} \tag{17}$$

which eventually yields the following expressions for the BLUE of $\boldsymbol{\beta}$

$$\begin{aligned} \boldsymbol{b} &= \widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{y}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{V}} & \widetilde{\boldsymbol{X}} \\ \widetilde{\boldsymbol{X}}' & \mathbf{0} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R}' \\ \boldsymbol{X}' & \boldsymbol{R} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{s} \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{18}$$

and for its dispersion matrix

$$\begin{aligned} \boldsymbol{V}(\boldsymbol{b}) &= \sigma^2 \widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{C}}' = -\sigma^2 \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{V}} & \widetilde{\boldsymbol{X}} \\ \widetilde{\boldsymbol{X}}' & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{I} \end{bmatrix} \\ &= -\sigma^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R}' \\ \boldsymbol{X}' & \boldsymbol{R} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{I} \end{bmatrix}. \end{aligned} \tag{19}$$

Straightforward application of standard partitioned inversion rules leads to the more convenient formulae

$$\boldsymbol{b} = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{R} \\ \boldsymbol{R}' & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} \\ \boldsymbol{s} \end{bmatrix}, \quad \boldsymbol{A} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X} \tag{20}$$

$$\boldsymbol{V}(\boldsymbol{b}) = \sigma^2 \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{R} \\ \boldsymbol{R}' & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}. \tag{21}$$

A second way to tackle the issue of finding the BLUE of $\boldsymbol{\beta}$ brings us back to unified least squares theory (Rao [9]). In this connection, observe that after (14) and (16), the matrix $\widetilde{\boldsymbol{C}}$ provides the minimum $\widetilde{\boldsymbol{V}}$-(semi)norm generalized inverse of $\widetilde{\boldsymbol{X}}$ (Rao and Mitra [15]). Then take

$$\boldsymbol{W} = \widetilde{\boldsymbol{V}} + \widetilde{\boldsymbol{X}}\boldsymbol{U}\widetilde{\boldsymbol{X}}' \tag{22}$$